CHEBYSHEV COLLOCATION METHOD ON SOLVING STABILITY OF A LIQUID LAYER FLOWING DOWN AN INCLINED PLANE

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SUMMARY

The Orr–Sommerfeld equation governing the linear stability of a liquid layer flowing down an inclined plane appears as a differential eigenvalue problem with the eigenvalue involved nonlinearly in the free surface boundary conditions which causes the problem more difficult to solve than most other Orr–Sommerfeld problems. A numerical solver based on Chebyshev spectral collocation discretization auxiliary with the companion matrix method is employed to solve the problem. The superiority in the accuracy of the solver is shown by its validation with other papers.

1. INTRODUCTION

The linear stability Orr–Sommerfeld problem of a viscous fluid layer flowing down an inclined plane under the action of gravity is re–solved here by a novel numerical approach–spectral collocation method. Since the paper focuses on the details of numerical method, the derivation of the governing differential equation, that is Orr–Sommerfeld equation, and the associated boundary conditions can be found in Yih[9] and Lin[8]. As to its physics, a recent review by Chang[3] describing the linear and nonlinear instabilities of a free–falling film is worth referring to.

Following Lin[8], the geometric configuration of the flow is shown in Fig. 1. with d denoting the Nusselt flat–film thickness, β the angle of inclination of the plane. Using the maximum velocity of the base flow $U_m = g \sin\beta / 2\nu$ (g is the gravitational acceleration, ν the kinematic viscosity) as the characteristic velocity and d the characteristic length, the dimensionless governing differential equation is the famous Orr–Sommerfeld equation,

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha Re[(U - c)(\phi'' - \alpha^2\phi) - U''\phi], \tag{1}$$

where α is the wavenumber, c the complex wave speed, $Re = U_m d/\nu$ the Reynolds number, U the dimensionless velocity profile of the base flow:

$$U(y) = 1 - y^2, \tag{2}$$

and ϕ a function of y representing the y–direction distribution of the perturbed streamfunction ψ by

$$\psi = \phi(y)\exp[i\alpha(x - ct)]. \tag{3}$$

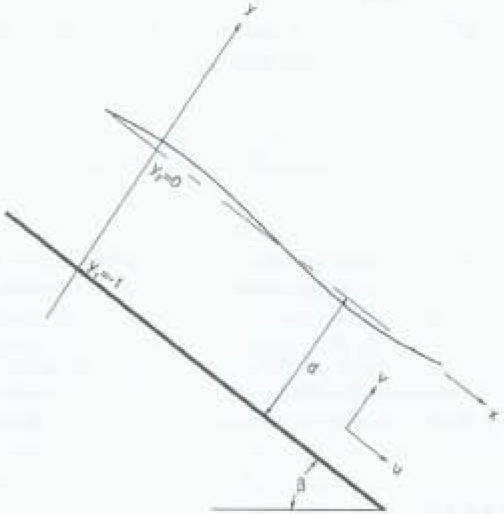

Figure 1: Definition sketch for a liquid layer flowing down an inclined plane.

The velocity perturbations u' and v' in the x and y directions can be expressed in terms of ψ by

$$u' = \partial\psi/\partial y, \tag{4}$$

$$v' = -\partial\psi/\partial x. \tag{5}$$

The boundary conditions are nonslip conditions at the bottom wall ($y = -1$),

$$\phi(-1) = \phi'(-1) = 0, \tag{6}$$

and the continuity of shear and normal stresses at the free surface ($y = 0$),

$$\phi''(0) + \left(\alpha^2 - \frac{2}{c'}\right)\phi(0) = 0, \tag{7}$$

$$-\left(\alpha\frac{2\cot\beta + \alpha^2 S Re}{c'}\right)\phi(0) + \alpha(Re\, c' + 3i\alpha)\phi'(0) - i\phi'''(0) = 0, \tag{8}$$

where $S = We^{-1} = T/\rho d U_m^2$ (We is the Weber number, T the surface tension, ρ the density) and $c' = c - 1$. Since S is dependent on U_m (and thus on flow rate), some papers prefer using the Kapitza number $\gamma = 3^{1/3}T/\rho\nu^{4/3}g^{1/3} = S Re^{5/3}(\frac{3}{2}\sin\beta)^{1/3}$ which is only a function of the physical properties of the liquid. Then, Eq.(8) is replaced by

$$-\left(\alpha\frac{2\cot\beta + \alpha^2\gamma Re^{-2/3}(\frac{3}{2}\sin\beta)^{-1/3}}{c'}\right)\phi(0) + \alpha(Re\, c' + 3i\alpha)\phi'(0)$$
$$-i\phi'''(0) = 0, \tag{8a}$$

Eq.(1), (6), (7) and (8) or (8a) are homogeneous and a nontrivial solution exists if there exists a relation between α, c, β, Re, and S (or γ):

$$f(\alpha, c, \beta, Re, S \text{ or } \gamma) = 0. \tag{9}$$

They construct a differential eigenvalue problem with the eigenvalue c and the associated eigenfunction ϕ to be solved. Although, the dimensionless velocity profile of the base flow is same as plane Poiseuille flow, the current Orr–Sommerfeld problem is much more difficult to solve than that for plane Poiseuille flow. It is because the boundary conditions Eq.(7) and (8) at free surface involve the eigenvalue nonlinearly. For temporal instability, c is complex ($c = c_r + ic_i$). Eq.(9) implies $c_r = c_r(\alpha, Re)$, $c_i = c_i(\alpha, Re)$ if β and S (or γ) are given. The flow is stable if $c_i < 0$ and unstable if $c_i > 0$, and $c_i(\alpha, Re) = 0$ gives a neutral stability curve on α–Re plane for given β and S (or γ).

There are two linear stability modes encountered in the current problem: surface mode (also named as soft mode) and shear mode (also named as hard mode). The surface mode is essentially surface wave driven by gravity-capillary effects slightly modified by viscosity, and the shear mode is basically the Tollmien–Schlichting wave modified by the presence of the free surface. The surface mode usually with long wavelength (compared with d) is the dominant unstable mode when Reynolds number is small to moderate ($1 < Re < 300$ suggested by Chang[3]), while the shear mode becomes dominant when Reynolds number is large ($Re > 1000$ suggested by Chang[3]) with wavelength comparable to or shorter than d.

Yih[9] solved the problem by perturbation expansion with the restriction to long waves (surface mode) and small Reynolds numbers, and obtained the famous result for the critical Reynolds number

$$Re_{cr} = \frac{5}{4}\cot\beta.$$

As to the shear mode, Lin[8] analytically solved the problem motivated by the Orr–Sommerfeld solutions of plane Poiseuille flow. However, Lin made a mistake on boundary condition Eq.(8) (the sign of the first term at the left-hand side of Eq.(8) is positive in Lin[8]), which is pointed out by De Bruin[5]. De Bruin uses Runge–Kutta integration to integrate Eq.(1) shooting for c with orthogonalization procedure to solve the problem with the correct boundary condition. The orthogonalization procedure is chiefly to keep particular solutions from depending on each other (caused by the stiffness of Eq.(1)) during the integration. Floryan et al.[6] following De Bruin's numerical method solved the problem with Eq.(8a) with nonzero surface tension extensively for both surface and shear modes. Chin et al.[4] also calculates the problem with Eq.(8a) by finite difference method with an additional consideration on the from factor effect on the velocity profile of the base flow. Ho and Patera[7] computes the problem with Eq.(8) using Hermitian finite element method as a preliminary study in their spectral element Navier–Stokes simulation of a free falling film.

The numerical methods used in the papers mentioned above are either Runge–Kutta integration shooting method or finite difference/element methods. The former can not calculate the whole Orr–Sommerfeld spectrum and is sensitive to the initial guess on c. Also, during iteration, the search on c complex plane can sometimes be tedious. The latter can calculate the whole Orr–Sommerfeld spectrum but the order of accuracy is low. In this paper, the novel spectral collocation method with much higher order of accuracy than conventional finite difference/element methods is employed to solve the whole Orr–Sommerfeld spectrum (for both surface and shear modes) from which the most unstable eigenvalue c is identified. Based on spectral collocation method, the solver is easily developed and highly efficient. The numerical procedure is derived in section 2 (numerical method) and the results are validated with Lin[7], De Bruin[5] and Floryan et al.[6] in section 3 (results and discussion).

2. NUMERICAL METHOD

The physical domain $y \in [-1, 0]$ needs to be mapped into the computational domain $z \in [-1, 1]$ by

$$y = \frac{z-1}{2}. \tag{10}$$

$\phi(y(z))$ is then expanded as a series of Chebyshev polynomials,

$$\phi(y(z)) = \sum_{k=0}^{N} a_k T_k(z). \tag{11}$$

The reason to choose Chebyshev polynomials among all kinds of orthogonal polynomials is that Chebyshev polynomials are orthogonal with respect to the weight function $1/(1-z^2)^{1/2}$ which underlies its emphasis near the boundaries and is particularly suitable to describe the boundary layer phenomenon frequently encountered in flow instability. Eq.(1), (6), (7) and (8) or (8a) can then be rewritten in terms of differential operators with respect to z:

$$\begin{aligned} L_4\phi &= cL_2\phi, & (12)\\ B_0\phi &= B_1\phi = 0, \quad \text{at} \quad z=-1, & (13)\\ E_2\phi &= cF_2\phi, \quad \text{at} \quad z=1, & (14)\\ E_3\phi &= cF_3\phi + c^2F_1\phi, \quad \text{at} \quad z=1, & (15) \end{aligned}$$

where

$$\begin{aligned} L_4 &= 16\,d^4/dz^4 - (8\alpha^2 + 4i\alpha Re\,U)\,d^2/dz^2 \\ &\quad + (\alpha^4 + i\alpha^3 Re\,U + i\alpha Re\,U''), & (16)\\ L_2 &= -4i\alpha Re\,d^2/dz^2 + i\alpha^3 Re, & (17)\\ B_0 &= 1, & (18)\\ B_1 &= d/dz, & (19)\\ E_2 &= 4\,d^2/dz^2 + (\alpha^2+2), & (20)\\ F_2 &= 4\,d^2/dz^2 + \alpha^2, & (21)\\ E_3 &= 8\,d^3/dz^3 - (6\alpha^2 + 2i\alpha Re)\,d/dz - i(2\alpha\cot\beta + \alpha^3 S Re), & (22)\\ F_3 &= 8\,d^3/dz^3 - (6\alpha^2 + 4i\alpha Re)\,d/dz, & (23)\\ F_1 &= 2i\alpha Re\,d/dz. & (24) \end{aligned}$$

or

$$\begin{aligned} E_3 &= 8\,d^3/dz^3 - (6\alpha^2 + 2i\alpha Re)\,d/dz - i(2\alpha\cot\beta \\ &\quad + \alpha^3\gamma Re^{-2/3}(\frac{3}{2}\sin\beta)^{-1/3}), & (22a) \end{aligned}$$

for boundary condition (8a).

Eq.(12)–(15) are to be collocated at Chebyshev–Gauss–Lobatto quadrature points (Canuto et al.[2]), distributed as

$$z_j = \cos\left(\frac{j\pi}{N}\right) \qquad j = 0, 1, 2, \ldots, N. \tag{25}$$

Let $\boldsymbol{\phi}$ with the entry $\phi_j = \phi(y(z_j))$ denotes the value of $\phi(y(z))$ at the collocation points in Eq.(25). d/dz is then approximated by the Chebyshev collocation derivative matrix $\boldsymbol{D}_N$ (Canuto et al.[2]):

$$(\boldsymbol{D}_N)_{ij} = \begin{cases} \frac{\bar{c}_i(-1)^{i+j}}{\bar{c}_j(z_i - z_j)}, & i \neq j, \\ \frac{-z_j}{2(1-z_j^2)}, & 1 \le i = j \le N-1, \\ \frac{2N^2+1}{6}, & i = j = 0, \\ -\frac{2N^2+1}{6}, & i = j = N, \end{cases} \tag{26}$$

and, likewise, d^n/dz^n is approximated by $\boldsymbol{D}_N{}^n$. The differential operators in Eq.(16)–(24) can then be transformed into matrix operators:

$$(\boldsymbol{L}_4)_{ij} = 16\,(\boldsymbol{D}_N{}^4)_{ij} - \left(8\alpha^2 + 4i\alpha Re\,U(z_i)\right)(\boldsymbol{D}_N{}^2)_{ij}$$

$$+ \left(\alpha^4 + i\alpha^3 Re\, U(z_i) + i\alpha Re\, U''(z_i)\right) \delta_{ij}, \tag{27}$$

$$(\boldsymbol{L}_2)_{ij} = -4i\alpha Re\, (\boldsymbol{D}_N{}^2)_{ij} + i\alpha^3 Re\, \delta_{ij}, \tag{28}$$

$$(\boldsymbol{B}_0)_{ij} = \delta_{ij}, \tag{29}$$

$$(\boldsymbol{B}_1)_{ij} = (\boldsymbol{D}_N)_{ij}, \tag{30}$$

$$(\boldsymbol{E}_2)_{ij} = 4\,(\boldsymbol{D}_N{}^2)_{ij} + (\alpha^2 + 2)\,\delta_{ij}, \tag{31}$$

$$(\boldsymbol{F}_2)_{ij} = 4\,(\boldsymbol{D}_N{}^2)_{ij} + \alpha^2 \delta_{ij}, \tag{32}$$

$$(\boldsymbol{E}_3)_{ij} = 8\,(\boldsymbol{D}_N{}^3)_{ij} - (6\alpha^2 + 2i\alpha Re)(\boldsymbol{D}_N)_{ij} - i(2\alpha \cot\beta + \alpha^3 S\, Re)\,\delta_{ij}, \tag{33}$$

$$(\boldsymbol{F}_3)_{ij} = 8\,(\boldsymbol{D}_N{}^3)_{ij} - (6\alpha^2 + 4i\alpha Re)(\boldsymbol{D}_N)_{ij}, \tag{34}$$

$$(\boldsymbol{F}_1)_{ij} = 2i\alpha Re\, (\boldsymbol{D}_N)_{ij}, \tag{35}$$

or

$$(\boldsymbol{E}_3)_{ij} = 8\,(\boldsymbol{D}_N{}^3)_{ij} - (6\alpha^2 + 2i\alpha Re)(\boldsymbol{D}_N)_{ij} - i(2\alpha \cot\beta + \alpha^3 \gamma Re^{-2/3} (\frac{3}{2}\sin\beta)^{-1/3})\,\delta_{ij}, \tag{33a}$$

for boundary condition (8a), where $\boldsymbol{L}_4, \ldots,$etc. denote the matrix operators approximating $L_4, \ldots,$etc. and δ_{ij} denotes Kronecker delta.

By Eq.(27)-(35), Eq.(12)-(15) are transformed into a nonlinear matrix generalized eigenvalue problem:

$$\boldsymbol{G}\phi = c\boldsymbol{H}\phi + c^2\boldsymbol{K}\phi, \tag{36}$$

where

$$(\boldsymbol{G})_{ij} = \left\{ \begin{array}{ll} (\boldsymbol{E}_3)_{0j}, & i = 0, \\ (\boldsymbol{E}_2)_{0j}, & i = 1, \\ (\boldsymbol{L}_4)_{ij}, & 2 \le i \le N-2, \\ (\boldsymbol{B}_0)_{Nj}, & i = N-1, \\ (\boldsymbol{B}_1)_{Nj}, & i = N, \end{array} \right\}, \quad 0 \le j \le N, \tag{37}$$

$$(\boldsymbol{H})_{ij} = \left\{ \begin{array}{ll} (\boldsymbol{F}_3)_{0j}, & i = 0, \\ (\boldsymbol{F}_2)_{0j}, & i = 1, \\ (\boldsymbol{L}_2)_{ij}, & 2 \le i \le N-2, \\ 0, & i = N-1,\, N, \end{array} \right\}, \quad 0 \le j \le N, \tag{38}$$

$$(\boldsymbol{K})_{ij} = \left\{ \begin{array}{ll} (\boldsymbol{F}_1)_{0j}, & i = 0, \\ 0, & 1 \le i \le N, \end{array} \right\}, \quad 0 \le j \le N. \tag{39}$$

Eq.(36) can be solved for the eigenvalue c satisfying

$$\det(\boldsymbol{G} - c\boldsymbol{H} - c^2\boldsymbol{K}) = 0. \tag{40}$$

Eq.(40) is numerically approximated by minimizing $|\det(\boldsymbol{G} - c\boldsymbol{H} - c^2\boldsymbol{K})|$, which can be performed by an IMSL routine UMINF based on a quasi-Newton method with an initial guess on $c = c_r + ic_i$. However, when N is large, the computer calculation of determinant overflows easily. A remedial alternative is to minimize the reciprocal of the condition number of the matrix rather than the determinant itself. The idea is that when the reciprocal of the condition number is close to 0, the matrix is extremely ill-conditioned and therefore nearly singular. Another problem in using IMSL routine UMINF is that a close initial guess on c, usually not available, is required for this minimization scheme to succeed. To cure this, Eq.(36) can be transformed into a linear matrix generalized eigenvalue problem by the companion matrix method (Bridges and Morris[1]):

$$\begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \phi \\ c\phi \end{bmatrix} = c \begin{bmatrix} \boldsymbol{H} & \boldsymbol{K} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \phi \\ c\phi \end{bmatrix}, \tag{41}$$

where $\boldsymbol{0}$ and $\boldsymbol{I}$ are the identity and null matrices with the same ranks as $\boldsymbol{G}, \boldsymbol{H}$ and $\boldsymbol{K}$. Eq.(41) can be directly solved by the IMSL routine GVCCG based on QZ algorithm. However, the matrix size in Eq.(41) is twice as large as Eq.(40), and the computation is not economic when N is large. Besides, its round-off error will accumulate to significant figures when N is very large. Hence, c is better first calculated by Eq.(41) with a smaller N, then this c is used as a good initial guess for Eq.(40) for further improvement with a larger N.

3. RESULTS AND DISCUSSION

Although the boundary condition (8) is mistaken in Lin[8], Lin's result is still worth comparing with for numerical validation. Using the incorrect boundary condition, the current solver recomputes a series of neutral c of shear mode in Lin[8], and the comparison with Lin's result is shown in TABLE I, in which the agreement is well.

The eigenvalue c under $\alpha = 1.062553, Re = 8126.813538$, in TABLE I is arbitrarily chosen to demonstrate the spectral accuracy of the current method by computing c against increasing N. The result is shown in TABLE II. The spectral accuracy can be more clearly observed from the diagram of the relative error vs. N in Figure 2 based on TABLE II. From Figure 2, the order of accuracy can reach as high as 33 when N is between 40 and 50.

The results in Floryan et al.[6] are also recalculated here for validation. The comparison is shown in TABLE III and IV. The agreement in TABLE III and IV is even better than TABLE I.

Table I: Current computation of shear mode eigenvalue c compared with Lin[8] with $S = 0, \beta = 1^\circ$.

α	Re	c	
		Lin [8]	current result, $N = 60$
0.371858	4.152194(5)	0.0706	$0.7055577(-1) + 0.3700788(-4)i$
0.539594	5.164135(4)	0.1295	$0.1294119 - 0.1004532(-3)i$
0.656387	1.976407(4)	0.1725	$0.1724954 - 0.3032945(-3)i$
0.849765	7327.351990	0.2367	$0.2372869 - 0.6205254(-3)i$
0.966308	5791.709595	0.2619	$0.2629749 - 0.3972811(-3)i$
1.039895	6476.701111	0.2640	$0.2651037 + 0.1893931(-3)i$
1.062553	8126.813538	0.2550	$0.2558536 + 0.5700734(-3)i$
1.060653	1.195320(4)	0.2370	$0.2374371 + 0.8192571(-3)i$
1.014058	2.406454(4)	0.2044	$0.2043330 + 0.7798818(-3)i$
0.946671	4.799068(4)	0.1750	$0.1747119 + 0.5498975(-3)i$
0.852935	1.138520(5)	0.1430	$0.1426459 + 0.2646038(-3)i$
0.736120	3.382576(5)	0.1100	$0.1097519 - 0.3297150(-4)i$
0.618260	1.141233(6)	0.0815	$0.8116823(-1) - 0.1218521(-3)i$

Table 2: eigenvalue c vs. N

N	c
20	$0.2560783 + 0.7455761(-3)i$
22	$0.2541468 + 0.1469023(-4)i$
24	$0.2565135 - 0.1089530(-2)i$
26	$0.2568238 + 0.1155089(-2)i$
28	$0.2555736 + 0.8622085(-3)i$
30	$0.2558768 + 0.4605091(-3)i$
32	$0.2558753 + 0.6403534(-3)i$
34	$0.2558131 + 0.5617639(-3)i$
36	$0.2558629 + 0.5570625(-3)i$
38	$0.2558553 + 0.5742978(-3)i$
40	$0.2558526 + 0.5694982(-3)i$
42	$0.2558540 + 0.5700405(-3)i$
44	$0.2558535 + 0.5701773(-3)i$
46	$0.2558536 + 0.5700383(-3)i$
48	$0.2558536 + 0.5700790(-3)i$
50	$0.2558536 + 0.5700739(-3)i$

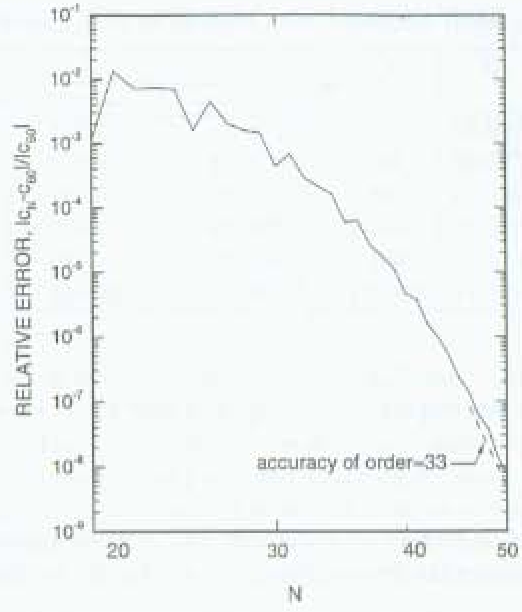

Figure 2: Fast decay of the relative error of c as N increases based on TABLE II.

Table 3: Current result compared with Floryan et al.[6]: surface modes with $\alpha = 0.27, \gamma = 4899.38, \beta = 4^\circ$.

Re	c	
	Floryan et al.[6]	current result, $N = 60$
1000	$1.09780 + 0.260614(-1)i$	$1.09978 + 0.260605(-1)i$
2000	$1.06222 + 0.203353(-1)i$	$1.06222 + 0.203355(-1)i$
5000	$1.03441 + 0.141295(-1)i$	$1.03441 + 0.141295(-1)i$
10000	$1.02241 + 0.105362(-1)i$	$1.02241 + 0.105362(-1)i$
40000	$1.00984 + 0.569076(-2)i$	$1.00984 + 0.569076(-2)i$
100000	$1.00582 + 0.373391(-2)i$	$1.00581 + 0.373666(-2)i$
1000000	$1.00163 + 0.125600(-2)i$	$1.00163 + 0.125521(-2)i$

Table 4: Current result compared with Floryan et al.[6]: neutral shear modes

β	γ	Re	α	c	
				Floryan et al.[6]	current result, $N = 60$
0.5′	0	8369.69	2.893	0.176474	$0.176474 - 0.224745(-6)i$
1′	100	5375.60	2.588	0.205978	$0.205977 - 0.258216(-6)i$
3′	500	3707.23	1.898	0.248637	$0.248637 - 0.104392(-7)i$
4′	1000	3829.26	1.691	0.254429	$0.254429 - 0.100337(-6)i$
1°	10000	5414.59	1.091	0.263420	$0.263420 - 0.163627(-7)i$
4°	20000	5498.74	1.074	0.263643	$0.263643 - 0.407821(-7)i$

Typical profiles of eigenfunction ϕ and its derivative ϕ' for shear and surface modes are shown respectively in Figure 3 and 4. The results agree well with De Bruin[4]. For shear mode, shown in Figure 3, ϕ and ϕ' resembles their counterparts of Tollmien–Schlichting wave for plane Poiseuille flow with slight modification at the free surface. For surface mode, shown in Figure 4, the magnitudes of ϕ and ϕ' are small near the wall but large as approaching the free surface, which indicates the general feature of gravity–driven free surface wave.

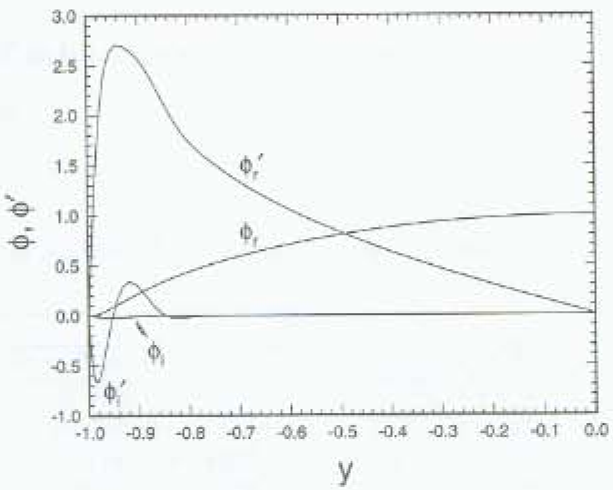

Figure 3: A shear mode eigenfunction ϕ and its derivative ϕ': $\alpha = 1.0354, c = 0.2044281 - 0.3341756(-4)i, Re = 24065, S = 0$, and $\beta = 1°$.

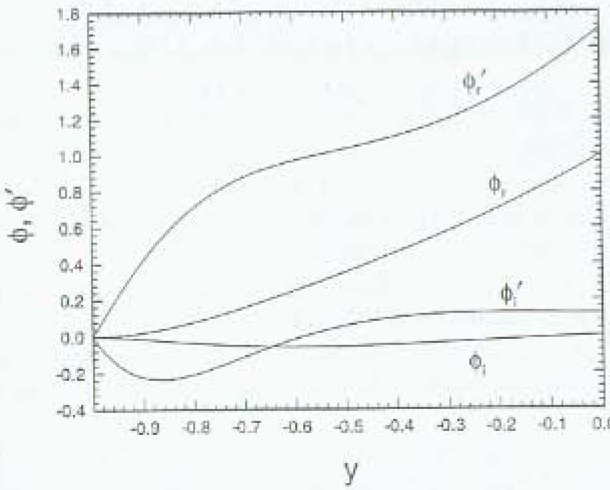

Figure 4: A surface mode eigenfunction ϕ and its derivative ϕ': $\alpha = 0.3597, c = 1.814062 + 0.3843755(-6)i, Re = 100, S = 0$, and $\beta = 1°$.

4. CONCLUSION

The current solver based on Chebyshev spectral collocation method together with the companion matrix method and the reciprocal of condition number minimization scheme shows high accuracy and computing efficiency in solving the whole Orr–Sommerfeld spectrum. Since the problem is physically better interpreted by the concept of convective instability which is actually observed in experiments, the future work is to solve the spatial version of the problem. It is more complicated and difficult to solve because the eigenvalue, that is the wavenumber α, appears nonlinearly up to α^4 in the governing equation. This also causes the rank of matrices involved in the companion matrix method twice as large as the current situation, which is definitely more cpu time consuming.

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